Question Sheet 2, Limits II. v1. 2020-21

Limits not existing

In the next two questions we look at examples of functions that do **not** have a limit at a point.

1. i) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

   Prove that the limit $\lim_{x\to 0} f(x)$ does not exist.

   Hint use proof by contradiction and the fact that in any interval we can find a rational number and an irrational number.

   ii) Prove that for all $a \in \mathbb{R}$ the limit $\lim_{x\to a} f(x)$ does not exist.

2. i) Show by means of an example that $\lim_{x\to a} \{f(x) + g(x)\}$ may exist even though neither $\lim_{x\to a} f(x)$ or $\lim_{x\to a} g(x)$ exist.

   ii) Do the same for $\lim_{x\to a} f(x)\, g(x)$.

   Hint: Construct f and g from a function in the previous question or a similar one seen in the notes.

One-sided limits

In the next two questions we examine a limit at a point by examining the two one-sided limits at that point.

3. Let

$$F(x) = \frac{x^2 - 1}{|x - 1|}.$$

   i) For what x is this well-defined?

   Hint Recall that $|y| = y$ if $y \geq 0$ and $= -y$ if $y < 0$.

   ii) Find $\lim_{x\to 1^+} F(x)$

iii) Find $\lim_{x\to 1^-} F(x)$.

iv) Does $\lim_{x\to 1} F(x)$ exist?

v) Sketch the graph of $F(x)\,, x \in \mathbb{R}$.

4. Sketch the graph of

$$f(x) = \begin{cases} 8 - x^2, & x < 2 \\ 3, & x = 2 \\ x^2 - 2 & x > 2. \end{cases}$$

Use the ε - δ definition to evaluate the following one-sided limits.

$$\text{i)} \lim_{x\to 2-} f(x) \qquad \text{and} \qquad \text{ii)} \lim_{x\to 2+} f(x)\,.$$

Does $\lim_{x\to 2} f(x)$ exist?

Limits at Infinity

In the next five questions we look at limits as $x \to +\infty$ and $x \to -\infty$.

5. Verify the ε - X definition of

$$\lim_{x\to+\infty} \frac{3x+3}{x-2} = 3.$$

6. Verify the ε - X definition of

$$\lim_{x\to-\infty} \frac{2x-2}{x+2} = 2.$$

7. Find the value of

$$\lim_{x\to+\infty} \frac{2-x^2}{x^2+2}$$

and show your value satisfies the ε - X definition.

8. Find the value of

$$\lim_{x\to-\infty} \frac{3x+3}{x-2},$$

and show your value satisfies the ε - X definition.

Extra questions for practice

9. Verify the ε - X definition of

$$\lim_{x\to+\infty} \frac{-2-x^2}{x^2-2} = -1.$$

10. Find the value of

$$\lim_{x\to-\infty} \frac{-2-x^2}{x^2-2},$$

and show your value satisfies the ε - X definition.